

PRINCIPAL GRADIENT SCHEMES HAVE REGULAR REDUCED CLOSED SUBSCHEMES

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1. INTRODUCTION

This paper represents the first steps in a program whose goal is to understand the formal properties of *gradient schemes*, i.e., schemes that are locally analytically cut out by the gradient of a function (see Definition 4.3). In [1], Clemens has shown that Hilbert Schemes of curves on K -trivial threefolds are gradient schemes. Therefore it is the hope that an understanding of gradient schemes will shed light on the geometry of these Hilbert schemes. The problem of understanding gradient schemes is also interesting from the point of view of commutative algebra in that we are trying to determine which ideals in power series rings are gradient ideals.

The contents of the paper are as follows. In Section 2 we review some basic algebraic facts and definitions about Jacobian ideals and gradient ideals in power series rings. In Section 3 we prove a regularity criterion (Theorem 3.3) for the reduced quotient ring of a power series ring modulo a principal ideal. This criterion is stated in terms of the associated primes of the Jacobian ideal and is interesting in its own right. We are currently working on generalizations. Finally, in Section 4 we prove our main result (Theorem 4.6), which states that principal gradient schemes have regular reduced subschemes. We consider complete intersection gradient schemes in [5].

1.1. Acknowledgments. I am indebted to Herb Clemens for his guidance and for suggesting this project. I would also like to thank Linda Chen and Sheldon Katz for valuable conversations. Finally, I would to thank Yu-Han Liu who has shown me an elementary proof of Theorem 4.6 that does not use Theorem 3.3.

2. BASIC FACTS REGARDING GRADIENT AND JACOBIAN IDEALS

Let k be a field of characteristic zero and consider the ring of formal power series $k[[x_1, \dots, x_n]]$. We gather some basic facts and definitions regarding Jacobian and gradient ideals in $k[[x_1, \dots, x_n]]$.

Definition 2.1. Let $f \in k[[x_1, \dots, x_n]]$ be a power series. The *gradient ideal* of f is the ideal

$$I_{\nabla f} := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \subseteq k[[x_1, \dots, x_n]],$$

Date: November 20, 2008.

and the *Jacobian ideal* of f is the ideal

$$(f, I_{\nabla f}) \subset k[[x_1, \dots, x_n]].$$

Remark 2.2. We may also write

$$\frac{\partial f}{\partial x_i}$$

as f_{x_i} .

Definition 2.3. Let $f \in k[[x_1, \dots, x_n]]$ be a nonzero power series. The *order* of f is the minimal total degree of all nonzero monomials appearing in f .

Lemma 2.4. *If $f \in k[[x_1, \dots, x_n]]$ is a squarefree power series, then the ring*

$$\frac{k[[x_1, \dots, x_n]]}{(f, I_{\nabla f})}$$

has dimension less than $n - 1$.

Proof. If

$$\dim \frac{k[[x_1, \dots, x_n]]}{(f, I_{\nabla f})} = n - 1,$$

then there exists an irreducible factor h of f such that for all $1 \leq i \leq n$ the power series h divides the partial derivative of f with respect to x_i . Write $f = h^k g$ where h does not divide g . Then for all $1 \leq i \leq n$ the factor h divides

$$\frac{\partial f}{\partial x_i} = h^k \frac{\partial g}{\partial x_i} + k h^{k-1} g \frac{\partial h}{\partial x_i}.$$

It follows that h divides all its partial derivatives, but this is impossible since there must exist an $1 \leq i \leq n$ such that the order of the series

$$\frac{\partial h}{\partial x_i}$$

is less than the order of h . □

Lemma 2.5. *Let h be an irreducible factor of a power series $f \in k[[x_1, \dots, x_n]]$. If for all $1 \leq i \leq n$ we have*

$$h^k \text{ divides } \frac{\partial f}{\partial x_i},$$

then h^{k+1} divides f .

Proof. Write $f = h^l g$, where h does not divide g . We wish to show that $l \geq k + 1$. For $1 \leq i \leq n$ we compute

$$\frac{\partial f}{\partial x_i} = h^{l-1} \left(h \frac{\partial g}{\partial x_i} + l \frac{\partial h}{\partial x_i} g \right). \quad (2.1)$$

If $l < k + 1$ then since h^k divides the right hand side in (2.1), it must be the case that h divides

$$\left(h \frac{\partial g}{\partial x_i} + l \frac{\partial h}{\partial x_i} g \right).$$

Since h does not divide g , it follows that h must divide all of its partials. This contradicts Lemma 2.4. □

Lemma 2.6. *Let $f \in k[[x_1, \dots, x_n]]$ be a power series that is not a unit. Then*

$$f \in \sqrt{I_{\nabla} f}.$$

Proof. This follows from the stronger statement that f is in the integral closure of the ideal $(x_1, \dots, x_k)I_{\nabla} f$. See [3]. \square

3. A REGULARITY CRITERION

We recall the Jacobian criterion for power series rings due to Nagata, see [2, Proposition 22.7.2]. For any k -algebra A we denote by $\text{Der}_k(A)$ the A -module of all k -linear derivations from A to A .

Theorem 3.1 (Nagata). *Let I be an ideal of the ring $k[[x_1, \dots, x_n]]$, and let \mathfrak{p} be a prime ideal with $I \subseteq \mathfrak{p}$. Then the ring*

$$\frac{k[[x_1, \dots, x_n]]_{\mathfrak{p}}}{I \cdot k[[x_1, \dots, x_n]]_{\mathfrak{p}}}$$

is regular if and only if there exist elements $g_1, \dots, g_k \in I$ and derivations $D_1, \dots, D_k \in \text{Der}_k(k[[x_1, \dots, x_n]])$ such that

- (1) *the images of g_1, \dots, g_k in $I \cdot k[[x_1, \dots, x_n]]_{\mathfrak{p}}$ generate $I \cdot k[[x_1, \dots, x_n]]_{\mathfrak{p}}$, and*
- (2) *$\det\{D_i g_j\} \notin \mathfrak{p}$.*

Corollary 3.2. *Let $f \in k[[x_1, \dots, x_n]]$ be an element that is not a unit. Then the local ring*

$$\frac{k[[x_1, \dots, x_n]]}{(f)}$$

is regular if and only if the ideal

$$(f, I_{\nabla} f) \tag{3.1}$$

is the unit ideal.

Proof. First, suppose that the ideal (3.1) is the unit ideal. Since f is not a unit, this means that one of the partial derivatives of f is a unit. We may assume that

$$\frac{\partial f}{\partial x_1} \in k[[x_1, \dots, x_n]]^{\times}.$$

But then the derivation $\frac{\partial}{\partial x_1}$ and the element f satisfy the conditions of Theorem 3.1, and

$$\frac{k[[x_1, \dots, x_n]]}{(f)}$$

is regular.

Now suppose that the ring

$$\frac{k[[x_1, \dots, x_n]]}{(f)}$$

is regular. Let $g_1, \dots, g_k \in k[[x_1, \dots, x_n]]$ and $D_1, \dots, D_k \in \text{Der}_k(k[[x_1, \dots, x_n]])$ be elements satisfying the conditions of Theorem 3.1. Since $\det\{D_i g_j\}$ is a unit, it

follows that $D_l g_m$ must be a unit for some l and m . Recall ([4, Theorem 30.6]) that $\text{Der}_k(k[[x_1, \dots, x_n]])$ is a free $k[[x_1, \dots, x_n]]$ -module of rank two with basis

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}.$$

Suppose

$$D_l = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} \quad (3.2)$$

for power series $\alpha_1, \dots, \alpha_n \in k[[x_1, \dots, x_n]]$. Since $D_l g_m$ is a unit, one of the summands in (3.2) applied to g_m is a unit. We may assume that

$$\frac{\partial g_m}{\partial x_1} \in k[[x_1, \dots, x_n]]^\times.$$

But $f \mid g_m$ so we may write $g_m = fh$ for some $h \in k[[x_1, \dots, x_n]]$. Applying the product rule for partial differentiation, we find

$$\frac{\partial g_m}{\partial x_1} = f \frac{\partial h}{\partial x_1} + h \frac{\partial f}{\partial x_1}.$$

Now f is not a unit by assumption so it follows that

$$\frac{\partial f}{\partial x_1} \in k[[x_1, \dots, x_n]]^\times$$

as required. \square

We next establish a Jacobian criterion that can determine when the reduced quotient ring of a ring is regular. Recall that if M is an R -module, we say that a prime ideal $\mathfrak{p} \subseteq R$ is an *associated prime* of M if $\mathfrak{p} = \text{Ann}(Rm)$ for some $m \in M$. The set of all associated primes of an R -module M is called $\text{Ass}_R(M)$. Recall that minimal elements of $\text{Ass}_R(0)$ correspond to irreducible components of $\text{Spec } R$ and non-minimal elements correspond to *embedded components*. Note that $\mathfrak{p} \in \text{Ass}_R(M)$ if and only if there exists an injective R -linear map

$$\frac{R}{\mathfrak{p}} \longrightarrow M. \quad (3.3)$$

Theorem 3.3. *Let f be an element of the ring $k[[x_1, \dots, x_n]]$ that is not a unit. Then the local ring*

$$\frac{k[[x_1, \dots, x_n]]}{\sqrt{(f)}}$$

is regular if and only if all elements of the set

$$\text{Ass}_{k[[x_1, \dots, x_n]]} \left(\frac{k[[x_1, \dots, x_n]]}{(f, I_{\nabla f})} \right),$$

have height one.

Proof. First suppose that $\frac{k[[x_1, \dots, x_n]]}{\sqrt{(f)}}$ is regular. It follows that $f = g^k$ for some irreducible element $g \in k[[x_1, \dots, x_n]]$. For if f were divisible by two distinct irreducible factors g and h not unit multiples of each other, then the quotient ring $\frac{k[[x_1, \dots, x_n]]}{\sqrt{(f)}}$ would have the cosets of g and h as zero-divisors. But this would contradict the fact that regular local rings are integral domains [4, Theorem 14.3]. Since $\sqrt{(f)} = (g)$, we find that the ideal $(g, I_{\nabla} g)$ is the unit ideal by Corollary 3.2. Hence,

$$\begin{aligned} \frac{k[[x_1, \dots, x_n]]}{(f, I_{\nabla} f)} &= \frac{k[[x_1, \dots, x_n]]}{(g^k, kg^{k-1}g_{x_1}, \dots, kg^{k-1}g_{x_n})} \\ &= \frac{k[[x_1, \dots, x_n]]}{(g^{k-1}(g, I_{\nabla} g))} \\ &= \frac{k[[x_1, \dots, x_n]]}{(g^{k-1})}, \end{aligned}$$

and

$$\text{Ass}_{k[[x_1, \dots, x_n]]} \left(\frac{k[[x_1, \dots, x_n]]}{(g^{k-1})} \right) = \begin{cases} \emptyset & \text{if } k = 1 \\ (g) & \text{if } k > 1 \end{cases}$$

as required.

Now, suppose that the ring

$$\frac{k[[x_1, \dots, x_n]]}{\sqrt{(f)}}$$

is not regular. We may assume that

$$f = \prod_{i=1}^k g_i^{e_i}$$

for some positive integers e_1, \dots, e_k and where the elements g_1, \dots, g_k are pairwise relatively prime irreducible factors of f . Then the radical of the ideal (f) is given by

$$\sqrt{(f)} = \left(\prod_{i=1}^k g_i \right).$$

We then may compute the Jacobian ideal of f :

$$\begin{aligned} (f, I_{\nabla} f) &= \left(\prod_{i=1}^k g_i^{e_i}, \prod_{i=1}^k g_i^{e_i} \left(\sum_{j=1}^n \frac{e^j g_{i x_j}}{g_i} \right), \dots, \prod_{i=1}^k g_i^{e_i} \left(\sum_{j=1}^n \frac{e^j g_{i x_n}}{g_i} \right) \right) \\ &= \left(\prod_{i=1}^k g_i^{e_i-1} \right) \left(\prod_{i=1}^k g_i, \prod_{i=1}^k g_i \left(\sum_{j=1}^n \frac{e^j g_{i x_j}}{g_i} \right), \dots, \prod_{i=1}^k g_i \left(\sum_{j=1}^n \frac{e^j g_{i x_n}}{g_i} \right) \right). \end{aligned}$$

Since g_i cannot divide all of its partial derivatives by Lemma 2.4, it follows that the ideal

$$\left(\prod_{i=1}^k g_i, \prod_{i=1}^k g_i \left(\sum_{j=1}^n \frac{e^j g_{i x_j}}{g_i} \right), \dots, \prod_{i=1}^k g_i \left(\sum_{j=1}^n \frac{e^j g_{i x_n}}{g_i} \right) \right) \quad (3.4)$$

is either the unit ideal or has height greater than two. An ideal

$$(\phi_1, \dots, \phi_s) \subseteq k[[x_1, \dots, x_n]]$$

is the unit ideal if and only if ϕ_i is a unit for some $1 \leq i \leq s$, and a power series $\psi \in k[[x_1, \dots, x_n]]$ is a unit if and only if its constant term is not zero. For $1 \leq j \leq n$, the constant term of

$$\prod_{i=1}^k g_i \left(\sum_{i=1}^k \frac{e^i g_{ix_j}}{g_i} \right)$$

is the sum of the constant terms of the elements

$$e^i g_i \cdots g_{i-1} g_{ix_j} g_{i+1} \cdots g_k \quad (3.5)$$

for $1 \leq i \leq k$. If $k \geq 2$, the constant term in (3.5) is zero and the ideal in (3.4) is not the unit ideal. Hence, the ideal (3.4) has height two if $k \geq 2$, and the theorem follows. If $k = 1$, we put $g_1 = g$ and $e_1 = e$ and compute

$$\begin{aligned} (f, I_{\nabla f}) &= (g^e, e g^{e-1} g_{x_1}, \dots, e g^{e-1} g_{x_n}) \\ &= (g^{e-1})(g, I_{\nabla g}), \end{aligned}$$

and $(g, I_{\nabla g})$ is not the unit ideal by Corollary 3.2 because we are assuming that the ring

$$\frac{k[[x_1, \dots, x_n]]}{\sqrt{(f)}} = \frac{k[[x_1, \dots, x_n]]}{(g)}$$

is not regular. Since g is irreducible, Lemma 2.4 implies that

$$\dim \frac{k[[x_1, \dots, x_n]]}{(g, I_{\nabla g})} < n - 1.$$

Hence, there exists a prime ideal

$$\mathfrak{p} \in \text{Ass}_{k[[x_1, \dots, x_n]]} \left(\frac{k[[x_1, \dots, x_n]]}{(g, I_{\nabla g})} \right)$$

such that \mathfrak{p} has height two or more.

We are now in the situation where

$$\frac{k[[x_1, \dots, x_n]]}{(f, I_{\nabla f})} = \frac{k[[x_1, \dots, x_n]]}{(\psi) \cdot I}$$

for some element $\psi \in k[[x_1, \dots, x_n]]$ and some ideal I such that

$$\mathfrak{p} \in \text{Ass}_{k[[x_1, \dots, x_n]]} \left(\frac{k[[x_1, \dots, x_n]]}{I} \right)$$

As in (3.3), there is an injective $k[[x_1, \dots, x_n]]$ -linear map

$$\frac{k[[x_1, \dots, x_n]]}{\mathfrak{p}} \longrightarrow \frac{k[[x_1, \dots, x_n]]}{I}.$$

Composing with the injective $k[[x_1, \dots, x_n]]$ -linear map

$$\frac{k[[x_1, \dots, x_n]]}{I} \longrightarrow \frac{k[[x_1, \dots, x_n]]}{(\psi) \cdot I}$$

induced by multiplication by the element ψ shows that

$$\mathfrak{p} \in \text{Ass}_k[[x_1, \dots, x_n]] \left(\frac{k[[x_1, \dots, x_n]]}{(\psi) \cdot I} \right)$$

as required. \square

4. APPLICATION TO GRADIENT IDEALS AND GRADIENT SCHEMES

Definition 4.1. Let I be an ideal of the power series ring $k[[x_1, \dots, x_n]]$. We say that I is a *gradient ideal* if there exists an element $f \in k[[x_1, \dots, x_n]]$ such that $I = I_{\nabla f}$.

Lemma 4.2. *The property of an ideal $I \subseteq k[[x_1, \dots, x_n]]$ being a gradient ideal is invariant under isomorphism, the gradient ideal of an element being sent to the gradient ideal of the image of the element under the isomorphism.*

Proof. Let θ be an isomorphism

$$k[[x_1, \dots, x_n]] \xrightarrow{\theta} k[[u_1, \dots, u_n]], \quad (4.1)$$

and put $\theta(x_i) = x_i(u_1, \dots, u_n)$ for each $1 \leq i \leq n$. For $1 \leq i \leq n$, we apply the chain rule to find

$$\frac{\partial \theta(f)}{\partial u_i} = \sum_{j=1}^n \theta \left(\frac{\partial f}{\partial x_j} \right) \frac{\partial x_j}{\partial u_i}. \quad (4.2)$$

Since θ is an isomorphism, the Jacobian matrix of θ is invertible. It follows that the image of the gradient ideal of f under θ is equal to the gradient ideal of $\theta(f)$ as required. \square

Definition 4.3. Let X be a scheme of finite type over k , and let $P \in X$ be a k -rational point. We say that the pointed scheme (X, P) is a *gradient scheme* if the completion of the local ring at P with respect to its maximal ideal is isomorphic as a complete local k -algebra to

$$\frac{k[[x_1, \dots, x_n]]}{I}$$

for some gradient ideal $I \subseteq k[[x_1, \dots, x_n]]$. If I is a principal ideal, we say that the gradient scheme is *principal*.

We state a simple lemma regarding ideals in power series rings.

Lemma 4.4. *If an ideal $(f_1, \dots, f_k) \subseteq k[[x_1, \dots, x_n]]$ is principal, then*

$$(f_1, \dots, f_k) = (f_i)$$

for some $1 \leq i \leq n$.

For any ring R , let $\text{nil } R$ denote the nilradical of R .

Lemma 4.5. *Let (A, \mathfrak{m}) be a local k -algebra that is the localization of a finitely generated k -algebra, and let \hat{A} denote the \mathfrak{m} -adic completion of A . Then*

$$\left(\frac{A}{\text{nil } A} \right)^{\hat{}} \cong \frac{\hat{A}}{\text{nil } \hat{A}}.$$

Proof. Since completion is flat ([4, Theorem 8.8]) and $\widehat{I} = I \cdot \widehat{A}$ for any ideal $I \subseteq A$ ([4, Theorem 8.11]), we know that

$$\left(\frac{A}{\text{nil } A} \right)^\wedge \cong \frac{\widehat{A}}{(\text{nil } A) \cdot \widehat{A}}.$$

Since

$$\left(\frac{A}{\text{nil } A} \right)$$

is the localization of a finitely generated k -algebra and an integral domain, its completion has no nilpotent elements ([6, Ch. VIII, § 13, Theorem 32]). Hence, $(\text{nil } A) \cdot \widehat{A} = \text{nil } \widehat{A}$ as required. \square

Theorem 4.6. *If a gradient scheme (X, P) is principal, then its reduced subscheme is regular at P .*

Proof. Let \mathcal{O}_P denote the local ring at P . We must show that the local ring

$$\frac{\mathcal{O}_P}{\text{nil } \mathcal{O}_P}$$

is regular. A local ring is regular if and only if its completion is regular ([6, Ch. VIII, § 11]), so by Lemma 4.5 it suffices to show that the ring

$$\frac{\widehat{\mathcal{O}_P}}{\text{nil } \widehat{\mathcal{O}_P}}$$

is regular. We are assuming that the pointed scheme (X, P) is a principal gradient scheme so we may assume, by Lemma 4.4, that

$$\widehat{\mathcal{O}_P} \cong \frac{k[[x_1, \dots, x_n]]}{(f_{x_1})}$$

for some element $f \in k[[x_1, \dots, x_n]]$ that is not a unit and such that $f_{x_1} \neq 0$. To establish the theorem we must show that the ring

$$\frac{k[[x_1, \dots, x_n]]}{\sqrt{(f_{x_1})}}$$

is regular.

We proceed by analyzing the form of the element f . First note that if $f = g^k$ for some element $g \in k[[x_1, \dots, x_n]]$ having non-zero linear term, then the result follows from Lemma 4.2. Indeed, in this case there is a formal change of coordinates $(x_1, \dots, x_n) \mapsto (u_1, \dots, u_n)$ under which g^k is sent to u_1^k .

We next consider the case $f \in I_{\nabla f}$. In this case, we have $(f, I_{\nabla f}) = (f_{x_1})$ and hence all associated primes of the ring

$$\frac{k[[x_1, \dots, x_n]]}{(f, I_{\nabla f})} = \frac{k[[x_1, \dots, x_n]]}{(f_{x_1})}$$

have height one. It follows from Theorem 3.3 that the ring

$$\frac{k[[x_1, \dots, x_n]]}{\sqrt{(f)}}$$

is regular. Therefore $f = g^k$ for some power series g with nonzero linear term, and we are in the case of the previous paragraph.

To complete the proof it suffices to show that we must have $f \in (f_{x_1})$. By Lemma 2.6, we know that $f^k \in (f_{x_1})$ for some integer $k \geq 1$. This implies that if an irreducible power series h divides f_{x_1} it must also divide f . The result now follows from Lemma 2.5. \square

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